Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+y^2}+y^2}{xy}$. **SCORE: _____ / 35 PTS**

$$\underbrace{\left(x\sqrt{x^2+y^2}+y^2\right)}_{M} dx - \underbrace{xy}_{N}\, dy = 0$$ 3

$$M(tx,ty) = tx\sqrt{t^2x^2+t^2y^2} + t^2y^2 = t^2\left(x\sqrt{x^2+y^2}+y^2\right)$$

$$N(tx,ty) = -(tx)(ty) = t^2(-xy)$$

BOTH HOMOGENEOUS DEGREE 2 4

$y = vx$ 3

$$\left(x\sqrt{x^2+v^2x^2}+v^2x^2\right) dx - x^2v\,(v\,dx + x\,dv) = 0$$ 3 2 3

$$x^2\sqrt{1+v^2}\, dx = x^3 v\, dv$$

$$\frac{1}{x}\, dx = \frac{v}{\sqrt{1+v^2}}\, dv$$ 3

3 $$C + \ln|x| = \sqrt{1+v^2}$$ 5

$$= \sqrt{1+\frac{y^2}{x^2}}$$ 3

$$Cx + x\ln|x| = \sqrt{x^2+y^2}$$

$$y^2 = (Cx + x\ln|x|)^2 - x^2$$ 3

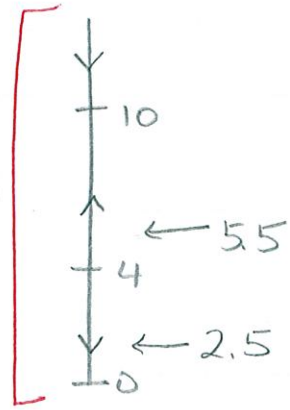
The number of people on an isolated island is changing at a rate of $\frac{dP}{dt} = -\frac{1}{200}(P^3 - 14P^2 + 40P)$ hundred people per year, where $P(t)$ is how many hundreds of people are on the island. Answer the following questions, and justify your answers using techniques discussed in class, **WITHOUT SOLVING THE DIFFERENTIAL EQUATION**.

SCORE: _____ / 25 PTS

[a] If there are currently 250 people on the island, what will be the population in the long term ?

$\frac{dP}{dt} = -\frac{1}{200}P(P-4)(P-10)$ 3

0 3



[b] If there are currently 550 people on the island, what will be the population in the long term ?

1000 3

[c] If there are currently 600 people on the island, what is the best estimate for the number of people on the island 1 month from now ?

$600 \text{ PEOPLE} + \left.\frac{dP}{dt}\right|_{P=6}$ HUNDRED PEOPLE/YEAR $*$ $\frac{1}{12}$ YEAR

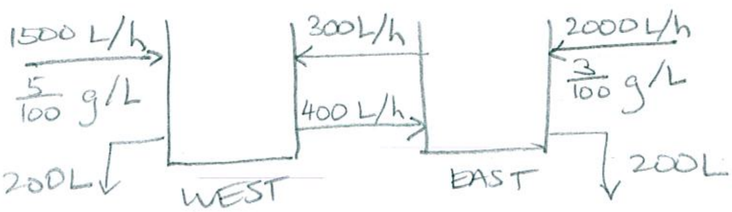
$= 600 + -\frac{1}{200}6(2)(-4)(100)\left(\frac{1}{12}\right)$ PEOPLE $= 602$ PEOPLE

3 2 2 3

A factory sits near two ponds, and is dumping its wastewater into both ponds. Every hour, the factory dumps 2000 liters of wastewater containing 3 grams of radioactive material per 100 liters into the east pond, and 1500 liters of wastewater containing 5 grams of radioactive material per 100 liters into the west pond. In addition, the two ponds are connected so that content from each pond seeps into the other. Each hour, 300 liters of the east pond's content seeps into the west pond, and 400 liters of the west pond's content seeps into the east pond. An additional 200 liters of each pond also drains away each hour.

SCORE: _____ / 20 PTS

If the east pond originally contained 50000 liters, the west pond originally contained 60000 liters, and both ponds were well-mixed at all times, write, **BUT DO NOT SOLVE**, a system of differential equations for the amount of radioactive material in each pond.



$V_{EAST}(t) = 50000 + (2400 - 500)t$
$= 50000 + 1900t$

$V_{WEST}(t) = 60000 + (1800 - 600)t$
$= 60000 + 1200t$

$$\frac{dE}{dt} = (2000)\left(\frac{3}{100}\right) + (400)\left(\frac{W}{60000+1200t}\right) - (500)\left(\frac{E}{50000+1900t}\right)$$

2 4 4

$$\frac{dW}{dt} = (1500)\left(\frac{5}{100}\right) + (300)\left(\frac{E}{50000+1900t}\right) - (600)\left(\frac{W}{60000+1200t}\right)$$

2 4 4

$$\frac{dE}{dt} = 60 + \frac{4W}{600+12t} - \frac{5E}{500+19t}, \quad \frac{dW}{dt} = 75 + \frac{3E}{500+19t} - \frac{6W}{600+12t}$$

Find a continuous solution of the initial value problem $(\tan x)\dfrac{dy}{dx} + 2y = \begin{cases} \sec x \csc x, & x < \frac{\pi}{3} \\ \sec x, & x > \frac{\pi}{3} \end{cases}$, $y(\frac{\pi}{4}) = -2$. **SCORE: _____ / 35 PTS**

3 $\dfrac{dy}{dx} + (2\cot x)y = \begin{cases} \csc^2 x, & x < \frac{\pi}{3} \\ \csc x, & x > \frac{\pi}{3} \end{cases}$ 3

$\mu = e^{\int 2\cot x\,dx} = e^{2\ln|\sin x|} = \sin^2 x$ 3

3 $(\sin^2 x)\dfrac{dy}{dx} + (2\sin x\cos x)y = \begin{cases} 1, & x < \frac{\pi}{3} \\ \sin x, & x > \frac{\pi}{3} \end{cases}$ 3

CHECK: $\dfrac{d}{dx}\sin^2 x = 2\sin x\cos x$ ✓ 2

2 $(\sin^2 x)y = \begin{cases} x + C, & x < \frac{\pi}{3} \\ -\cos x + D, & x > \frac{\pi}{3} \end{cases}$ 2

$y = \begin{cases} x\csc^2 x + C\csc^2 x, & x < \frac{\pi}{3} \\ -\csc x\cot x + D\csc^2 x, & x > \frac{\pi}{3} \end{cases}$ 3

$-2 = \frac{\pi}{4}(2) + C(2)$

$C = -1 - \frac{\pi}{4}$ 3

$\frac{\pi}{3}\left(\frac{4}{3}\right) + \left(-1 - \frac{\pi}{4}\right)\left(\frac{4}{3}\right) = -\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{3} + D\left(\frac{4}{3}\right)$

$D = \frac{\pi}{12} - \frac{1}{2}$ 5

$y = \begin{cases} (x - 1 - \frac{\pi}{4})\csc^2 x, & x < \frac{\pi}{3} \\ -\csc x\cot x + (\frac{\pi}{12} - \frac{1}{2})\csc^2 x, & x > \frac{\pi}{3} \end{cases}$ 3

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{ye^{2x}}{3y^2 - 2e^{2x}}$.

SCORE: _____ / 35 PTS

$$\underbrace{ye^{2x}}_{M}\,dx + \underbrace{(2e^{2x} - 3y^2)}_{N}\,dy = 0 \quad ③$$

$$M_y = e^{2x} \quad ② \qquad N_x = 4e^{2x} \quad ②$$

$$\frac{N_x - M_y}{M} = \frac{3e^{2x}}{ye^{2x}} = \frac{3}{y} \quad ③ \qquad \mu = e^{\int \frac{3}{y} dy} = e^{3\ln|y|} = y^3 \quad ③$$

$$\underbrace{y^4 e^{2x}}_{M}\,dx + \underbrace{(2y^3e^{2x} - 3y^5)}_{N}\,dy = 0 \quad ③$$

$$② \quad M_y = 4y^3 e^{2x} = N_x \quad \text{EXACT}$$

$$f = \int y^4 e^{2x}\,dx = \tfrac{1}{2}y^4 e^{2x} \; ② + C(y) \; ②$$

$$f_y = 2y^3e^{2x} \; ② + C'(y) = 2y^3e^{2x} - 3y^5 \quad ③$$

MUST BE FUNCTION OF ONLY y

$$C(y) = -\tfrac{1}{2}y^6 \quad ②$$

$$② \quad \tfrac{1}{2}y^4e^{2x} - \tfrac{1}{2}y^6 = C \quad ②$$

$$y^4 e^{2x} - y^6 = C \quad ②$$